Classifying prime character degree graphs

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Introduction

We operate under the following convention:

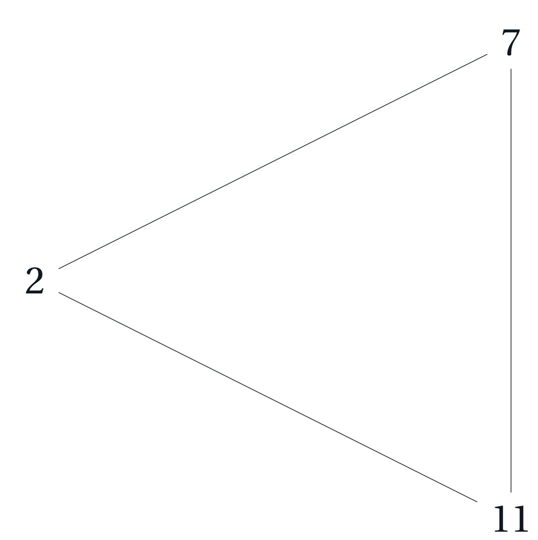
- G is a finite solvable group
- $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G
- $\mathrm{cd}(G) = \{\chi(1) \,:\, \chi \in \mathrm{Irr}(G)\}$
- $\Delta(G)$ is the prime character degree graph of G
- $\rho(G)$ denotes the vertices of $\Delta(G)$, which consists of all such prime divisors of $\mathrm{cd}(G)$
- There is an edge between two distinct vertices $p, q \in \rho(G)$ if there exists some character $a \in \mathrm{cd}(G)$ such that $pq \mid a$
- $\Delta(G)$ is a simple graph (so no direction on edges, no multiple edges, no loops on a vertex)

Claim 1. *Given any G, we can find $\Delta(G)$.*

Example 2. Let G be such that $\mathrm{cd}(G) = \{1, 6\}$.

Example 3. Let G be such that $\mathrm{cd}(G) = \{1, 2, 4, 8, 16, 32, 45\}$.

Example 4. Let G be such that $\mathrm{cd}(G) = \{1, 14, 22, 77\}$.



Goal

As stated in Claim 1, we can always find the prime character degree graph $\Delta(G)$ for any group G. This isn't exciting.
In our project, we essentially want to do the opposite. That is, given any graph, we want to decide if there is some group G so that its prime character degree graph $\Delta(G)$ is the same as the graph we started with. Specifically, we will work with a family of graphs constructed in a particular way.

Construction

Let $1 \le n \le k$. We now construct the family of graphs denoted by $\{\Sigma^*_{k,n}\}$. For subgraphs A and B, and a fixed vertex c, we have that:

(i) A is a complete graph on k vertices $a_1, a_2, \ldots, a_k$,
(ii) B is a complete graph on $k+n$ vertices $b_1, b_2, \ldots, b_k, \ldots, b_{k+n}$,
(iii) $c \notin \rho(A)$ and $c \notin \rho(B)$,
(iv) $\rho(A) \cap \rho(B) = \varnothing$,
(v) there is an edge between a_i and b_i for all $1 \le i \le k$,
(vi) there is an edge between a_i and b_{k+i} for all $1 \le i \le n$,
(vii) there is an edge between c and a_i for all $1 \le i \le k$ in $\Sigma^L_{k,n}$,
(viii) there is an edge between c and b_i for all $1 \le i \le k+n$ in $\Sigma^R_{k,n}$,
(ix) there are no edges in the graph $\Sigma^*_{k,n}$ other than the edges described in (i)–(viii).

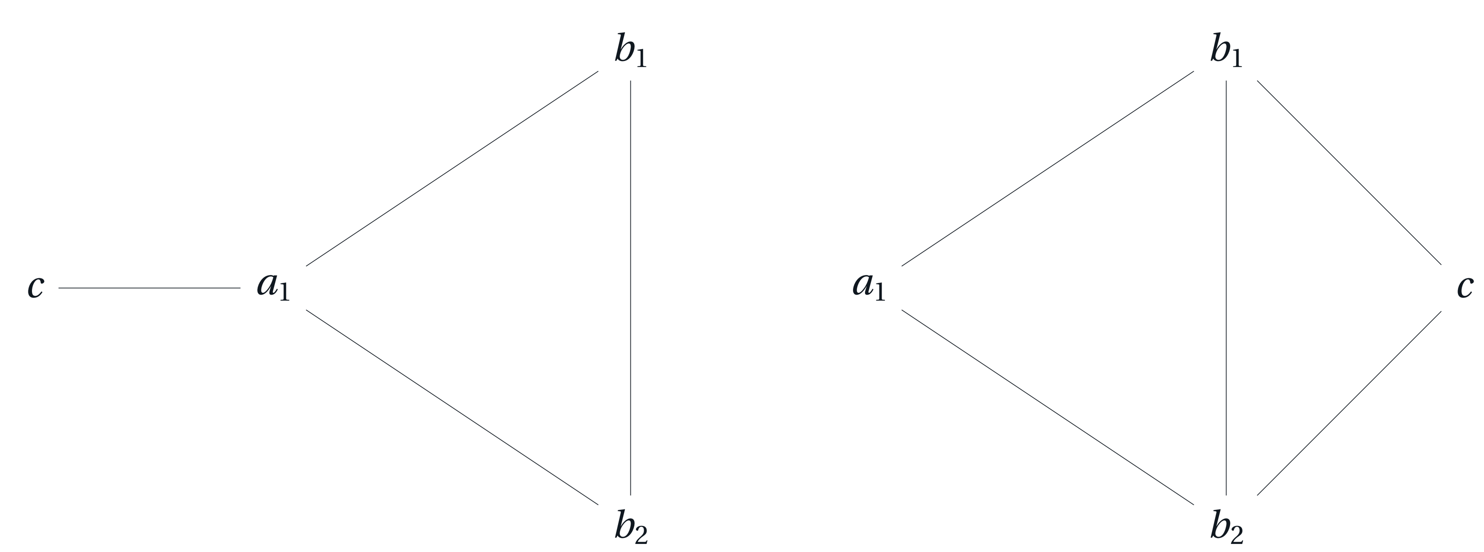
The notation of the graph $\Sigma^*_{k,n}$:

- The graph is read from left to right, with A on the left and B on the right
- There are two variants, $\Sigma^L_{k,n}$ and $\Sigma^R_{k,n}$, where the superscript tracks which side the fixed vertex c resides
- The subscript k represents how many vertices are in A
- The subscript n represents how many one-to-two distinct edge mappings exist from A to B

Main Result

Considering the above construction, we have the following:

Theorem 5. *The graphs $\Sigma^L_{1,1}$ and $\Sigma^R_{1,1}$ occur as the prime character degree graph of a solvable group (see* Figure 1*). The graphs $\Sigma^R_{2,1}$ and $\Sigma^R_{2,2}$ possibly occur as the prime character degree graph of a solvable group (see* Figure 2 *and* Figure 3*). Otherwise $\Sigma^*_{k,n}$ does not occur as the prime character degree graph of any solvable group.*



The graph $\Sigma^L_{1,1}$ The graph $\Sigma^R_{1,1}$

Figure 1

Proof. We proceed by induction on n. In particular, we handle the graphs $\Sigma^L_{k,n}$ and $\Sigma^R_{k,n}$ separately, and taking $k \ge 3$, we show that $\Sigma^*_{k,n}$ does not occur. All other cases with a small number of vertices are easily handled separately. The argument heavily relies on knowing information about subgraphs of the graph we're investigating. □

We want to take a closer look at the graphs that are yet-to-be classified. Specifically, the two graphs: $\Sigma^R_{2,1}$ and $\Sigma^R_{2,2}$.

The First Graph

We first consider the graph $\Sigma^R_{2,1}$.

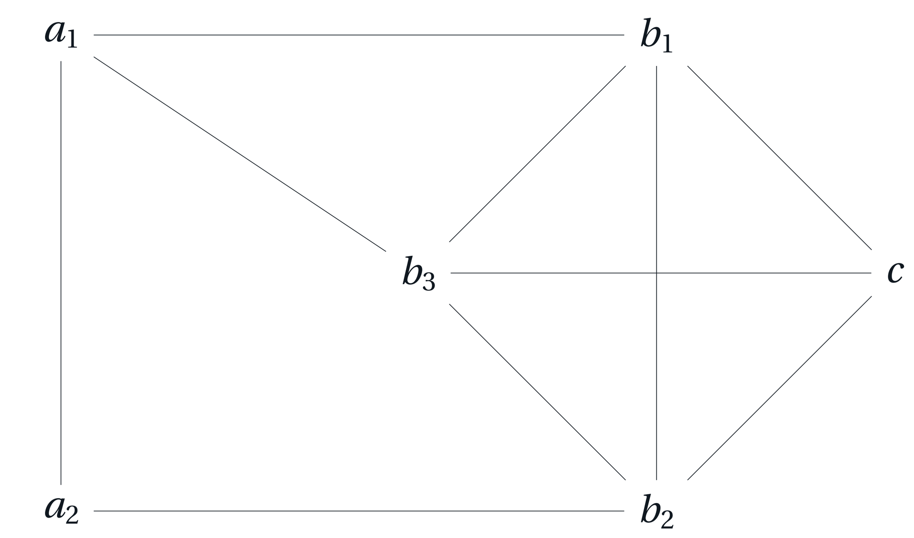


Figure 2

What went wrong:

- Deleting the vertex a_2 and all incident edges, along with the edge between b_1 and b_2, yields a graph with five vertices that is currently unknown as to whether it does or does not occur as the prime character degree graph.

The Second Graph

Next we consider the graph $\Sigma^R_{2,2}$.

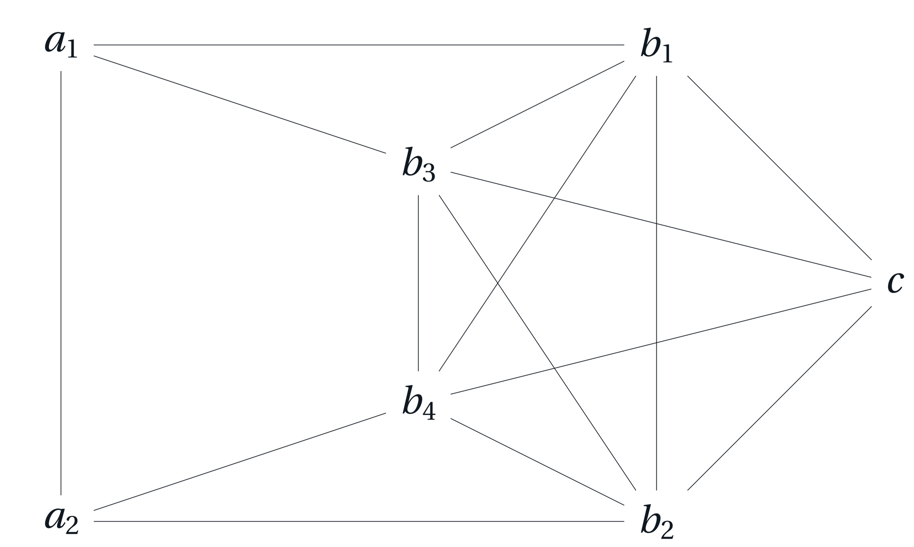


Figure 3

What went wrong:

- The disconnected graph with 2 vertices and 5 vertices occurs, and our strategy used for Theorem 5 required the disconnected graph to not occur.
- Deleting the vertex b_4 and all incident edges yields the graph in Figure 2, so we once again don't know information about one of the subgraphs.

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